

In: Einstein and Hilbert: Dark Matter ISBN 978-1-61324-840-9
 Editor: V. V. Dvoeglazov, pp. 19-36 © 2011 Nova Science Publishers, Inc.

The Principle of Solidarity: Geometrical Description of Interactions

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Abstract

We discuss the possibility of geometrizing interactions by exploiting the “principle of solidarity” between space-time and the physical phenomena occurring in it (formulated by the Italian mathematician B. Finzi in 1955). This is accomplished by means of a deformation of the Minkowski metric, implemented by assuming that the metric coefficients depend on the energy of the process considered. Such a formalism (“Deformed Special Relativity”) allows one, among the others, to deal with the breakdown of Lorentz invariance and to recover it in a generalized sense.

1. Introduction: The Finzi Principle of Solidarity

In 1955 the Italian mathematician Bruno Finzi, in his contribution to the book “*Fifty Years of Relativity*”[1], stated his “*Principle of Solidarity*”(PS)¹, that

¹It's quite difficult to express in English in a simple way the Italian words “*solidarietà*” and “*solidale*”, used by Finzi to mean the feedback between space-time and interactions. A possible way to render them is to use “*solidarity*” and “*solidly connected*”, respectively - at the price of partially loosing the common root of the Italian words - , with the warning that what Finzi

sounds “*It’s (indeed) necessary to consider space-time TO BE SOLIDLY CONNECTED with the physical phenomena occurring in it, so that its features and its very nature do change with the features and the nature of those. In this way not only (as in classical and special-relativistic physics) space-time properties affect phenomena, but reciprocally phenomena do affect space-time properties. One thus recognizes in such an appealing “Principle of Solidarity” between phenomena and space-time that characteristic of mutual dependence between entities, which is peculiar to modern science.*” Moreover, referring to a generic N-dimensional space: “*It can, a priori, be pseudoeuclidean, Riemannian, non-Riemannian. But — he wonders — how is indeed the space-time where physical phenomena take place? Pseudoeuclidean, Riemannian, non-Riemannian, according to their nature, as requested by the principle of solidarity between space-time and phenomena occurring in it.*”

Of course, Finzi’s main purpose was to apply such a principle to Einstein’s Theory of General Relativity, namely to the class of gravitational phenomena. However, its formulation is as general as possible, so to apply in principle to all the known physical interactions. Therefore, Finzi’s PS is at the very ground of any attempt at geometrizing physics, i.e. describing physical forces in terms of the geometrical structure of space-time.

Such a project (pioneered by Einstein himself) revealed itself unsuccessful even when only two interactions were known, the electromagnetic and the gravitational one. It was fully abandoned starting from the middle of the XXth century, due to the discovery of the two nuclear interactions, the weak and the strong one (apart from recent attempts based on string theory).

The basic problem is how to implement Finzi’s Principle of Solidarity for all interactions on a mere geometrical basis. Since, from an historical point of view, General Relativity (GR) is the only successful theoretical realization of geometrizing an interaction (the gravitational one), it is usually believed that the goal of geometrization of interactions can only be achieved by the tools of Riemannian spaces or of their suitable generalizations.

We want instead to show that implementing the Finzi principle can be obtained in the mere framework of Special Relativity, provided its very foundations are taken into proper account and suitably exploited. To this aim, let us analyze Special Relativity from an axiomatic standpoint.

really means is that the very structure of space-time is determined by the physical phenomena which do take place in it.

2. An Axiomatic View to Special Relativity

Special Relativity (SR) is essentially grounded on the properties of space-time, *i.e.* isotropy of space and homogeneity of space and time (as a consequence of the equivalence of inertial frames) and on the principle of relativity.

The two basic postulates of SR in its axiomatic formulation are [2]:

- 1 - *Space-time properties*: Space and time are homogeneous and space is isotropic.
- 2 - *Principle of Relativity (PR)*: All physical laws must be covariant when passing from an inertial reference frame K to another frame K' , moving with constant velocity relative to K .

The second postulate can be traced back to Galilei himself, who of course enunciated and applied it with reference to the laws of classical mechanics (the only ones known at his times). In fact, the Relativity Principle contains implicitly (somewhat hidden, but actually easily understood after a moment's thought) the basic point that, for a correct formulation of SR, *it is necessary to specify the total class, C_T , of the physical phenomena to which the PR applies*. The importance of such a specification is easily seen if one thinks that, from an axiomatic viewpoint, the only difference between Galilean and Einsteinian relativities just consists in the choice of C_T (*i.e.* the class of mechanical phenomena in the former case, and of mechanical and electromagnetic phenomena in the latter).

It is possible to show that, from the above two postulates, there follow — without any additional hypothesis — all the usual “principles” of SR, *i.e.* the “principle of reciprocity”, the linearity of transformations between inertial frames, and the invariance of light speed in vacuum.

Concerning this last point, it can be shown in general that postulates 1 and 2 above imply the existence of an invariant, real quantity, having the dimensions of the square of a speed, whose value must be experimentally determined in the framework of the total class C_T of the physical phenomena². Such an invariant speed depends on the interaction (fundamental, or at least phenomenological) ruling the physical phenomenon considered. Therefore *there is, a priori, an invariant speed for every interaction*, namely, a maximal causal speed for every interaction.

All the formal machinery of SR in the Einsteinian sense (including Lorentz transformations and their implications, and the metric structure of space-time)

²The invariant speed is obviously ∞ for Galilei's relativity, and c (light speed in vacuum) for Einstein's relativity.

is simply a consequence of the above two postulates and of the choice, for the total class of physical phenomena C_T , of the class of mechanical and electromagnetic phenomena.

If different explicit choices of C_T are made, one gets a priori different realizations of the theory of relativity (in its abstract sense), each one embedded in the previous. Of course, the principle of relativity, together with the specification of the total class of phenomena considered, necessarily entails in all cases, for consistency, the uniqueness of the transformation equations connecting inertial reference frames³.

The attempt at including the class of nuclear and subnuclear phenomena in the total class of phenomena for which Special Relativity holds true is therefore expected to imply a generalization of the Minkowski metric, analogously to the generalization from the Euclidean to the Minkowski metric in going from mechanics to electrodynamics.

However, in order to avoid misunderstandings, it must be stressed that such an analogy with the extension of the Euclidean metric has to be understood not in the purely geometric meaning, but rather in the sense (as already stressed by Penrose [3]) of Euclidean geometry as a physical theory.

Indeed, the generalized metric must be equipped with a dynamic character and be not only a consequence, but also an effective description of (the interaction involved in) the class of phenomena considered. This allows one in this way to get a feedback between interactions and space-time structure, already accomplished for gravitation in General Relativity.

This complies with the "*Principle of Solidarity*" stated by Finzi in the form already quoted above, which can be embodied in the following third principle of Relativity:

3 - *Principle of Solidarity (PS)*: Each class of phenomena (namely, each interaction) determines its own space-time.

The fundamental problem is now: *How to endow the metric of the Minkowski space-time with a geometrical structure able to describe the interaction involved in a given process?* We will answer this question in the following.

³The hypothesis of the existence *a priori* of different relativities for different interactions — formulated by Recami and one of the present authors (R.M.) on the basis of the above critical analysis of the foundations of Special Relativity — can be considered a generalization of the point of view advocated by Lorentz, according to which different interactions require different coordinate transformations between inertial reference frames.

3. Energy and the Finzi Principle

At present, General Relativity (GR) is the only successful theoretical realization of geometrizing an interaction (the gravitational one). As is well known, energy plays a fundamental role in GR, since the energy-momentum tensor of a given system is the very source of the gravitational field.

A moment's thought shows that this occurs actually also for other interactions. Let us remind, for instance, the case of Euclidean geometry in its intrinsic meaning of a theory of physical reality at its basic classical (macroscopic) level. In fact, it describes in a quantitative way, in mathematical language, the relations among measured physical entities — distances, in this case —, and therefore the physical space in which phenomena occur.

However, the measurement of distances depends on the motion of the body which actually performs the measurement. Such a dependence is indeed not on the *kind* of motion, but rather on the *energy* needed to let the body move, and on the *interaction* providing such energy. The measurement of time needs as well a periodic motion with constant frequency, and therefore it too depends on the energy and on the interaction.

This simple example shows how *energy does play a fundamental role in determining the very geometrical structure of space-time* (in analogy with the General-Relativistic case, where — as already noted — the energy-momentum tensor is the source of the gravitational field). Let us stress that such a viewpoint is very similar, on many respects, to the Ehlers-Pirani-Schild scheme [4] (based on the earlier work of Weyl), in which the geometry of space-time is operationally determined by using the trajectories of free-falling objects (geodesics). In this framework, the points of space-time become physically real in virtue of the geometrical relations between them, and the classical particle motion is exploited to obtain the geometry of space-time (the argument can be extended to quantum motion as well [5]).

Generalizing such an argument, we can state that exchanging energy between particles amounts to measure operationally their space-time separation. Of course such a process depends on the interaction involved in the energy exchange; moreover, each exchange occurs at the maximal causal speed characteristic of the given interaction. It is therefore natural to assume that the measurement of distances, performed by the energy exchange according to a given interaction, realizes the “solidarity principle” between space-time and interactions at the microscopic scale.

By starting from such considerations, a possible way to implement Finzi's

principle for *all* fundamental interactions is provided by the formalism of *Deformed Special Relativity (DSR)* developed in the last decade of the XX century. It is based on a *deformation* of the Minkowski space, namely a space-time endowed with a metric whose coefficients just depend on energy (in the sense specified later on). Such an energy-dependent metric does assume a *dynamic role*, thus providing a geometrical description of the fundamental interaction considered and implementing the feedback between space-time structure and physical interactions which is just the content and the heritage of Finzi's principle.

The generalization of the Minkowski space implies, among the others, new, generalized transformation laws, which admit, as a suitable limit, the Lorentz transformations (just like Lorentz transformations represent a covering of the Galilei-Newton transformations) [6].

Then, the solidarity principle allows one to recover the basic features of the relativity theory in the Lorentz (not Einstein) view (Lorentzian relativity), namely different interactions entail different coordinate transformations and different invariant speeds.

4. Description of Interactions by Energy-Dependent Metrics

We will now show how the dynamic role of the energy, in describing the structure of space-time, can be exploited in order to geometrize all four fundamental interactions, so to comply with the Finzi principle. As already stressed above, this can be achieved by suitably *deforming* space-time, according to what dictated by the energy involved in the process, ruled by the interaction considered. Speaking in a figurative language, we can say that in such a view spacetime is not a rigid (and passive) background, but a sort of elastic carpet, able to change its shape according to the (energy of) the interaction involved, and to react in turn on the process, thus affecting its dynamics in a active way.

4.1. Deformed Minkowski Space-Time

In the attempt at a geometrical implementation of the Finzi principle, we have therefore to take into account the role of energy in determining an interaction, and the different “relativities” obtained in correspondence to different classes of physical phenomena.

As is well known, the Minkowski metric ⁴

$$g = \text{diag}(1, -1, -1, -1) \quad (1)$$

is a generalization of the Euclidean metric $\varepsilon = \text{diag}(1, 1, 1)$. By the considerations of the previous sections, we can assume that the metric describes, in an effective way, the interaction, and that there exist interactions more general than the electromagnetic ones (which, as well known, are long-range and derivable from a potential).

The simplest generalization of the space-time metric which accounts for such more general properties of interactions is provided by a *deformation*, η , of the Minkowski metric (1), defined as [6]

$$\eta = \text{diag}(b_0^2, -b_1^2, -b_2^2, -b_3^2). \quad (2)$$

Of course, from a formal point of view metric (2) is not new at all. Deformed Minkowski metrics of the same type have already been proposed in the past in various physical frameworks, starting from Finsler's generalization of Riemannian geometry [7] to Bogoslawski's anisotropic space-time [8] (just based on a Finslerian metric) to the isotopic Minkowski space [9]. A phenomenological deformation of the type (2) was also obtained by Nielsen and Picek [10] in the context of the electroweak theory. Moreover, although for quite different purposes, "quantum" deformed Minkowski spaces have been also considered in the context of quantum groups [11]. Leaving to later considerations the true specification of the exact meaning of the deformed metric (2) in our framework, let us right now stress two basic points.

1 - Firstly, metric (2) is supposed to hold at a *local* (and not global) scale, *i.e.* to be valid not everywhere, but only in a suitable (local) space-time region (characteristic of both the system and the interaction considered). We shall therefore refer often to it as a "*topical*" deformed metric⁵.

⁴In the following, lower Latin indices take the values $\{1, 2, 3\}$ and label spatial dimensions, whereas lower Greek indices vary in the range $\{0, 1, 2, 3\}$, with 0 referring to the time dimension. For brevity's sake, we shall denote simply by x the (contravariant) four-vector (x^0, x^1, x^2, x^3) . Moreover, we adopt the signature $(+, -, -, -)$ for the four-dimensional space-time, and employ the notation "*ESC on*" ("*ESC off*") to mean that the Einstein sum convention on repeated indices is (is not) used.

⁵Notice that the assumed local validity of (2) differentiates this approach from those based on Finsler's geometry or from the Bogoslawski's one (which, at least in their standard meaning, do consider deformed metrics at a *global* scale), and makes it similar, on some aspects, to the

In the present case, the term ‘local’ must be understood in the sense that a deformed metric of the kind (2) describes the geometry of a 4-dimensional variety attached at a point x of the standard Minkowski space-time, in the same way as a local Lorentz frame is associated (as a tangent space) to each point of the (globally Riemannian) space of Einstein’s GR. Another example, on some respects more similar to the present formalism, is provided by a space-time endowed with a vector fibre-bundle structure, where a Riemann space with constant curvature is attached at each point x .

2 - Secondly, metric (2) is regarded to play a *dynamic role*. So, in order to comply with the solidarity principle, we assume that the parameters $b_\mu (\mu = 0, 1, 2, 3)$ are, in general, real and positive functions of a given set of observables $\{O\}$ characterizing the system (in particular, of its total energy exchange, as specified later):

$$\{b_\mu\} = \{b_\mu(\{O\})\} \in R_0^+, \forall \{O\} \quad (3)$$

The set $\{O\}$ represents therefore, in general, a set of non-metric variables ($\{x_{n.m.}\}$).

Eq. (2) therefore becomes:

$$\begin{aligned} \eta_{\mu\nu} &= \eta_{\mu\nu}(\{O\}) \\ &= \text{diag}(b_0^2(\{O\}), -b_1^2(\{O\}), -b_2^2(\{O\}), -b_3^2(\{O\})). \end{aligned} \quad (4)$$

However, for the moment the deformation of the Minkowski space will be discussed only from a formal point of view, by disregarding the problem of the observables on which the coefficients b_μ actually depend (it will be faced later on).

It is now possible to define a generalized (“*deformed*”) Minkowski space $\tilde{M}(x, \eta(\{O\}))$ with the same local coordinates x of M (the four-vectors of the

philosophy and methods of the isotopic generalizations of Minkowski spaces [9]. However, it is well known that Lie-isotopic theories rely in an essential way, from the mathematical standpoint, on (and are strictly characterized by) the very existence of the so-called isotopic unit. In the following, such a formal device will not be exploited (because unessential on all respects), so that the present formalism is not an isotopic one. Moreover, from a physical point of view, the isotopic formalism is expected to apply only to strong interactions. On the contrary, it will be assumed that the (effective) representation of interactions through the deformed metric (2) does hold for *all* kinds of interactions (at least for their nonlocal component). In spite of such basic differences this formalism shares some common formal results — as we shall see in the following — with isotopic relativity (like the mathematical expression of the generalized Lorentz transformations: see [6].)

usual Minkowski space), but with metric given by the metric tensor η (4). The generalized interval in \tilde{M} is therefore given by $(x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$, with c being the usual light speed in vacuum) (ESC on) [6]:

$$\begin{aligned} ds^2(\{O\}) &\equiv b_0^2(\{O\})c^2dt^2 - b_1^2(\{O\})(dx^1)^2 - b_2^2(\{O\})(dx^2)^2 - b_3^2(\{O\})(dx^3)^2 \\ &= \eta_{\mu\nu}(\{O\})dx^\mu dx^\nu = dx * dx. \end{aligned} \quad (5)$$

The last step in (5) defines the scalar product $*$ in the deformed Minkowski space \tilde{M} .

It is worth to recall that the deformation of the metric, resulting in the interval (5), represents a geometrization of a suitable space-time region (corresponding to the physical system considered) that describes, in the average, the effect of nonlocal interactions on a test particle. It is clear that there exist infinitely many deformations of the Minkowski space (precisely, ∞^4), corresponding to the different possible choices of the parameters b_μ , a priori different for each physical system.

Moreover, since the usual, “flat” Minkowski metric g (1) is related in an essential way to the electromagnetic interaction, it must be understood that electromagnetic interactions imply the presence of a fully Minkowskian metric⁶.

Once the mathematical body of our formalism is specified, one has now to give a physical soul to it, in order to comply with the Finzi principle. On the basis of the discussion of Sect.3, *we have to take, as observable O on which the metric coefficients $b_\mu(\{O\})$ depend, the total energy E exchanged by the physical system considered during the interaction process:*

$$\{O\} \equiv E \Leftrightarrow \{b_\mu(\{O\})\} \equiv \{b_\mu(E)\}, \quad \forall \mu = 0, 1, 2, 3. \quad (6)$$

Actually, since all the functions $\{b_\mu\}$ are dimensionless, they must depend on a dimensionless variable. Then, one has to divide the energy E by a constant E_0 (in general characteristic of each fundamental interaction), with dimensions of energy, so that:

$$\{b_\mu(\{O\})\} \equiv \{b_\mu(E/E_0)\}, \quad \forall \mu = 0, 1, 2, 3. \quad (7)$$

⁶Actually, a deformed metric of the type (4) is required if one wants to account for possible nonlocal electromagnetic effects (see [6]).

Thus, the distance measurement is accomplished by means of the deformed metric tensor function of the energy, given explicitly by

$$\eta_{\mu\nu}(E) = \text{diag}(b_0^2(E), -b_1^2(E), -b_2^2(E), -b_3^2(E)).$$

Any interaction can be therefore phenomenologically described by metric (8) in an *effective* way. This is true in general, but necessary in the case of non-local and nonpotential interactions. For force fields which admit a potential, such a description is complementary to the actual one.

One is therefore led to put forward a revision of the concept of “geometrization of an interaction”: each interaction produces its own metric, formally expressed by the metric tensor (8), but realized via different choices of the set of parameters $b_\mu(E)$. Otherwise said, the $b_\mu(E)$ ’s are peculiar to every given interaction. The statement that (8) provides us with a metric description of an interaction must be just understood in such a sense.

Therefore, the energy-dependent deformation of the Minkowski metric implements a generalization of the concept of geometrization of an interaction (in accordance with Finzi’s principle). The GR theory implements a geometrization (at a *global* scale) of the gravitational interaction, based on its derivability from a potential and on the equivalence between the inertial mass of a body and its “gravitational charge”. The formalism of energy-dependent metrics allows one instead to implement a geometrization (at a *local* scale) of any kind of interaction, at least on a phenomenological basis. As already stressed before, such a formalism applies, in principle, to both fundamental and phenomenological interactions, either potential (gravitational, electromagnetic) or nonpotential (strong, weak), *local* and *nonlocal*, for which either an Equivalence Principle holds (as it is the case of gravitation) or (in the more general case) the inertial mass of the body *is not* in general proportional to its charge in the force field considered (e.m., strong, and weak interaction).

Let us explicitly stress that the theory of SR based on metric (4) has nothing to do with General Relativity. Indeed, in spite of the formal similarity between the interval (5), with the b_μ functions of the coordinates, and the metric structure of a Riemann space, in this framework no mention at all is made of the equivalence principle between mass and inertia, and among non-inertial, accelerated frames. Moreover, General Relativity describes geometrization on a large-scale basis, whereas the special relativity with topical deformed metric describes local (small-scale) deformations of the metric structure (although the term “small scale” must be referred to the real dimensions of the physical system considered). But the basic difference is provided by the fact that

actually the deformed Minkowski space \tilde{M} has zero curvature, as it is easily seen by remembering that, in a Riemann space, the scalar curvature is constructed from the derivatives, with respect to space-time coordinates, of the metric tensor. In other words, the space \tilde{M} is *intrinsically flat* — at least in a mathematical sense.

Namely, it would be possible, in principle, to find a change of coordinates, or a rescaling of the lengths, so as to recover the usual Minkowski space. However, such a possibility is only a mathematical, and not a physical one. This is related to the fact that the energy of the process is fixed, and cannot be changed at will. For that value of the energy, the metric coefficients do possess values different from unity, so that the corresponding space \tilde{M} , for the given energy value, is actually different from the Minkowski one. The usual space-time M is recovered for a special value E_0 of the energy (characteristic of any interaction), such that indeed

$$\eta(E_0) = g = \text{diag}(1, -1, -1, -1). \quad (8)$$

Such a value E_0 (which must be derived from the phenomenology) will be referred to as *the threshold energy of the interaction considered*. It can be seen that it is just the constant appearing in Eq. (7).

4.2. Energy as Dynamic Variable

The basic point of the present way of geometrizing an interaction (thus implementing the Finzi legacy) consists in an “upsetting” of the space-time-energy parametrization. Whereas for potential interactions there exists a potential energy depending on the space-time metric coordinates, one has here to deal with a deformed metric tensor η , whose coefficients depend on the energy, that thus assumes a *dynamic* role. However, the identification of energy as the physical observable on which the metric must depend leaves open the question, what energy? Let us answer this question.

From the physical point of view, E is the measured energy of the system, and thus a merely phenomenological variable. As is well known, all the present physically realizable detectors work *via* their electromagnetic interaction in the usual space-time M . This is why, in this formalism, the Minkowski space and the e.m. interaction do play a fundamental role. The former is — as already stressed — the cornerstone on which to build up the generalization of Special Relativity based on the deformed metric (8). The latter is the comparison term for all fundamental interactions. Let us recall that they are strictly

interrelated, since it is just electromagnetism which determines the Minkowski geometry. Then, stating that the measurement of E occurs *via* the e.m. interaction amounts to say that it is measured in M . This ensures that the total energy is conserved, due the validity of the Hamilton theorem in Minkowski space. In summary, *E has to be understood as the energy measured by the detectors through the e.m. interaction in Minkowskian conditions and under validity of total energy conservation.*

From the mathematical standpoint, E has to be considered as a dynamic variable, because it specifies the dynamic behavior of the process under consideration, and, through the metric coefficients, provides us with a dynamic map — in the energy range of interest — of the interaction ruling the given process.

Let us notice that metric (8) plays, for nonpotential interactions, a role analogous to that of the Hamiltonian H for a potential interaction. In particular, the metric tensor η as well is not an input of the theory, but must be built up from the experimental knowledge of the physical data of the system concerned (in analogy with the specification of the Hamiltonian of a potential system). However, there are some differences between η and H worth to be stressed. Indeed, as is well known, H represents the total energy E_{tot} of the system irrespective of the value of E_{tot} and the choice of the variables. On the contrary, $\eta(E)$ describes the variation in the measurements of space and time, in the physical system considered, as E_{tot} changes; therefore, η does depend on the numerical value of H , but not on its functional form. The explicit expression of η depends only on the interaction involved⁷.

One may be puzzled about the dependence of the metric on the energy, which is not an invariant under usual Lorentz transformations, but transforms like the time-component of a four vector.

Actually, energy has to be regarded, in this formalism, from two different points of view. One has, on one side, the energy as measured in full Minkowskian conditions, which, as such, behaves as a genuine four-vector under usual Lorentz transformations (in the sense that it changes in the usual way if we go, say, from the laboratory frame to another frame in uniform mo-

⁷It is worth recalling that the use of an energy-dependent space-time metric can be traced back to Einstein himself, who generalized the Minkowski interval as follows

$$ds^2 = \left(1 + \frac{2\phi}{c^2}\right) c^2 dt^2 - (dx^2 + dy^2 + dz^2)$$

(where ϕ is the Newtonian gravitational potential), in order to account for the modified rate of a clock in presence of a (weak) gravitational field.

tion with respect to it). Once fixed the frame, one gets a measured value of the energy for a given process. This is the value which enters, as a parameter, in the expression (8) of the deformed metric. Such an energy, therefore, is no longer to be considered as a four vector in the deformed Minkowski space, but it is just a quantity whose value determines the deformed geometry of the process considered (or, otherwise speaking, which selects the deformed space-time we have to use to describe the phenomenon)⁸.

The problem of a metric description of a given interaction is thus formally reduced to the determination of the coefficients $b_\mu(E)$ from the data on some physical system, whose dynamic behavior is ruled by the interaction considered.

4.3. Deformed Special Relativity

In order to develop the relativity theory in a deformed Minkowski space-time, one has to suitably generalize and clarify the basic concepts which are at the very foundation of SR.

Let us first of all define a “topical inertial frame”:

i) A *topical “inertial” frame* (TIF) is a reference frame in which space-time is homogeneous, but space is not necessarily isotropic.

Then, a “*generalized principle of relativity*”, or “*principle of metric invariance*”, can be stated as follows:

ii) all physical measurements within every topical “inertial” frame must be carried out via the *same* metric.

We named “*Deformed Special Relativity*” (DSR) [6] the generalization of SR based on the above two postulates, and whose space-time structure is given by the deformed Minkowski space \tilde{M} introduced in Sect. 2. Let us also warn the reader against confusing this formalism with a different generalization of SR, i.e. Doubly Special Relativity [12], that uses the same acronym. This latter theory is essentially based on the quantum deformation of the Poincaré algebra, precisely, its κ -deformation. In such a kind of deformation, one essentially modifies the commutation relations of the Poincaré generators, whereas in the DSR framework the deformation concerns primarily the metrical structure of the space-time (although the Poincaré algebra is affected, too: see [6]).

⁸This different view to energy constitutes the basic point to building up a five-dimensional space-time, in which E does just represent the extra dimension (see [6]).

However, it is not clear at present if the two theories may have some points in common (for instance, the energy dependence of the metric in position space).

Moreover, henceforth we shall use the notation g_{DSR} for the metric tensor of DSR (in order to distinguish it from — but also to stress its affinities with — the standard Minkowskian metric tensor $g \equiv g_{SR}$), so that (with reference to Eq. (8))

$$g_{\mu\nu,DSR}(E) = \text{diag}(b_0^2(E), -b_1^2(E), -b_2^2(E), -b_3^2(E)) \quad (9)$$

is the covariant deformed metric tensor of \tilde{M} .

The corresponding deformed interval is of course

$$\begin{aligned} ds^2(E) &= g_{\mu\nu,DSR}(E) dx^\mu dx^\nu \\ &= b_0^2(E) c^2 dt^2 - b_1^2(E) dx^2 - b_2^2(E) dy^2 - b_3^2(E) dz^2. \end{aligned} \quad (10)$$

In matrix notation, the deformed interval (11) reads

$$ds^2(E) = (dX)^T g_{DSR}(E) dX, \quad (11)$$

where dX is the 4×1 column vector with elements dx^μ (so that $(dX)^T = (dx^0 \ dx^1 \ dx^2 \ dx^3)$, with the upper "T" denoting matrix transposition), and $g_{DSR}(E)$ is the 4×4 matrix (10).

5. DSR and Lorentz Invariance Breakdown

Let us now discuss the link between DSR and the violation of local Lorentz Invariance (LLI).

Theoretical speculations on the validity of LLI and SR can be traced back to the mid of the XX century. These early works were based on the existence of an absolute object in vacuum (like e.g. an universal length) [13].

In recent times, there has been an increasing interest in theoretical formalisms admitting for LLI violation [14].

They can be roughly divided in two classes: unified theories and theories with modified spacetimes. To the former one belong e.g. Grand-Unified Theories, (Super) String/Brane theories, (Loop) Quantum Gravity, and the so-called "effective field theories". The latter include e.g. foam-like quantum spacetimes, spacetimes endowed with a nontrivial topology or with a discrete structure at the Planck length, κ -deformed Lie algebra noncommutative spacetimes (for instance Doubly Special Relativity [12]).

Although LLI breakdown has been also discussed within the framework of the Standard Model [15], an extension of the Standard Model has been proposed [16] by assuming that the breakdown of Lorentz and/or CPT invariance is due to spontaneous symmetry breaking (namely to a non-invariance of the vacuum under these symmetries).

More recently, it was shown by Bogoslovsky [17] that Lorentz symmetry violation might occur without violation of relativistic symmetry represented in such a case by the 3-parameter group of the so-called generalized Lorentz boosts. This group serves as a noncompact homogeneous subgroup of the 8-parameter isometry group of the flat Finslerian spacetime with partially broken 3D isotropy. Such event space, generalizing the Minkowski space, coincides with that firstly introduced by the same author [8]. From this work it follows, among the others, that the physical carrier of the space-time anisotropy is the anisotropic fermion-antifermion condensate, arising from spontaneous breaking of initial gauge symmetry (for instance, in the Standard Model). Later on the results obtained in this work were mostly reproduced with the help of the techniques of continuous deformations of the Lie algebras and nonlinear realizations [18]⁹. Such a formalism was called in [18] “General Very Special Relativity” [19], while the 8-parameter group of Finslerian isometries was called $\text{DISIM}_b(2)$, i.e. Deformed Inhomogeneous SIMilitude group, that includes the 2-parameter Abelian homogeneous noncompact subgroup. The $\text{DISIM}_b(2)$ invariant nonlinear Dirac equation was proposed in ref.[20].

Coming again to DSR, let us remark the mathematically self-evident, but physically basic, point that the generalized metric (10) (and the corresponding interval) is clearly *not preserved by the usual Lorentz transformations*. If Λ_{SR} is the 4×4 matrix representing a standard Lorentz transformation, this amounts to say that the similarity transformation generated by Λ_{SR} does not preserve the deformed metric tensor g_{DSR} :

$$(\Lambda_{SR})^T g_{DSR} \Lambda_{SR} \neq g_{DSR}. \quad (12)$$

This is by no means an unexpected result, at the light of the axiomatic formulation of Special Relativity (see Sect.2). However, as a consequence, the deformed metric structure of \tilde{M} *violates* the standard Lorentz invariance, characteristic of the usual Minkowski space-time M . In this sense, therefore, we can state that DSR is strictly related to (and able to describe) the possible

⁹However, in [18] a different notation was used in comparison with [17]. In particular, the parameter that characterizes the space anisotropy magnitude was designated by “ b ” instead of “ r ”.

breakdown of Lorentz invariance, since the deformed metrics are no longer kept invariant by the standard Lorentz transformations.

However, it is possible to construct *deformed Lorentz transformations*, (i.e. isometries of \tilde{M}) which *do preserve* the generalized metric and interval (10,11)¹⁰. Therefore, Lorentz invariance, broken by the energy-dependent deformation of the space time *in its usual sense*, namely as a special-relativistic symmetry property of the interactions and/or the physical systems, *is recovered*, in the framework of DSR, in a generalized, wider meaning. We shall name *deformed Lorentz invariance (DLI)* this extended LI.

The mathematical formulation of DLI is provided by the following equation

$$\Lambda_{DSR,int.}^T(E) g_{DSR,int.}(E) \Lambda_{DSR,int.}(E) = g_{DSR,int.}(E). \quad (13)$$

(where we emphasized the dependence of the deformed Lorentz transformations on the interaction considered). It can be read as follows:

- *For every physical interaction, which affects the space-time geometry by deforming it in a way described by the metric tensor $g_{DSR,int.}$, it is always possible to find deformed Lorentz transformations $\Lambda_{DSR,int.}$ preserving the deformed geometrical structure of space-time for the interaction considered, namely (from a mathematical point of view) generating similarity transformations which leave the deformed metric tensor invariant.*

Then, we can state that DSR not only permits to deal with LI breakdown on a physical basis, but allows one to recover Lorentz invariance as an extended, higher symmetry of physics, valid for systems and/or interactions violating LI according to the usual Special Relativity, in the usual Minkowski space-time.

In conclusion, we want to stress that the DSR formalism has a number of possible implications and developments, both from a theoretical and an experimental view. These are, among the others: the possibility of getting phenomenological metrics (derived from experimental data) for the four fundamental interactions, which evidence departures from Minkowski metric in suitable energy ranges; the extension of the formalism to a five-dimensional scheme of the Kaluza-Klein type; the prediction of new physical effects some of which experimentally verified [21], like the existence of piezonuclear reactions, namely nuclear reactions triggered by pressure in liquids and solids in non-Minkowskian conditions. We refer the reader to ref.[6] for further details.

¹⁰For their explicit form we refer the reader to refs.[6].

References

- [1] B. Finzi: “Relatività Generale e Teorie Unitarie”, in *Cinquant’anni di Relatività* (in Italian), ed. M. Pantaleo (Giunti, Firenze, Italy, 1955), pg. 194.
- [2] E. Recami and R. Mignani: *Riv. Nuovo Cimento* **4**, n.2 (1974), and references therein.
- [3] R. Penrose: *The Emperor’s New Mind* (Oxford University Press, 1989).
- [4] J. Ehlers, F.A.E. Pirani and A. Schild, in *Papers in Honour of J. L. Synge*, edited by L. O’Raifeartaigh (Clarendon Press, Oxford 1972).
- [5] J.S. Anandan: in *Potentiality, Entanglement and Passion-at-a-distance - Quantum Mechanical Studies for Abner Shimony*, vol. 2, edited by R. S. Cohen, M. Horne and J. Stachel (Kluwer, Dordrecht, Holland 1997), p. 31.
- [6] F. Cardone and R. Mignani: *Energy and Geometry - An Introduction to Deformed Special Relativity* (World Scientific Series in Contemporary Chemical Physics, vol. 22) (World Scientific, Singapore, 2004); *Deformed Spacetime - Geometrizing Interactions in Four and Five Dimensions* (Springer-Verlag, New York, 2007); and references therein.
- [7] For a review of Finsler geometry, see e.g. Z. Shen: *Lectures on Finsler Geometry* (World Scientific, Singapore, 2001).
- [8] G.Yu. Bogoslovsky: *Nuovo Cimento B* **40**, 99 (1977). For a review, see G.Yu. Bogoslovsky: *Fortsch. Phys.* **42**, 2 (1994).
- [9] The Lie-isotopic generalization of Special Relativity was introduced by R.M. Santilli in *Lett. Nuovo Cim.* **37**, 565 (1983). For a review of Lie-isotopic theories, see e.g. R.M. Santilli: *Found.Phys.* **27**, 625 (1997), and refs. therein..
- [10] H. Nielsen and I. Picek: *Nucl.Phys.* **B211**, 269 (1983).
- [11] See e.g. A. U. Klimyk and K. Schmudgen: *Quantum Groups and Their Representations* (Texts and Monographs in Physics) (Springer-Verlag, New York, 1997).

- [12] See *e.g.* G. Amelino-Camelia: *Int. J. Mod. Phys. D* **11**, 1643 (2002), and refs. therein.
- [13] For a review of these early attempts, see *e.g.* F. Cardone and R. Mignani: *Found. Phys.* **29**, 1735 (1999).
- [14] For a review, see D. Mattingly: “Modern Tests of Lorentz Invariance”, *Living Rev. Relativity*, 8, 5 (2005) [Online Article]: cited [7 Sept. 2005], <http://www.livingreviews.org/lrr-2005-5>.
- [15] S. Coleman and S.L. Glashow: *Phys. Rev. D* **59**, 116008 (1999); R. Jackiw: “Chern-Simons violation of Lorentz and PCT symmetries in electrodynamics” (hep-ph/9811322), 1998, and references therein.
- [16] See *e.g.* *CPT and Lorentz Symmetry I, II, III*, V.A. Kostelecky ed. (World Scientific, Singapore, 1999, 2002, 2004), and refs. therein.
- [17] G.Yu.Bogoslovsky: *Phys. Lett.A*, **350**, 5 (2006); *SIGMA*, **1**, 017 (2005).
- [18] G.Gibbons, J.Gomis and C.Pope: *Phys. Rev.D* **76**, 081701(R) (2007).
- [19] In this connection, see also A.G.Cohen and S.L.Glashow: *Phys. Rev. Lett.* **97**, 021601 (2006).
- [20] G.Yu.Bogoslovsky and H.F.Goenner: *Phys. Lett.A*, **323**, 40 (2004).
- [21] U. Bartocci, F. Cardone and R. Mignani: *Found. Phys. Lett.* **14**, 51 (2001); F. Cardone, R. Mignani, W. Perconti and R. Scrimaglio: *Phys. Lett. A* **326**, 1 (2004); F. Cardone, R. Mignani and A. Petrucci: *Phys. Lett. A* **373**, 1956 (2009); F. Cardone, G. Cherubini, R. Mignani, W. Perconti, A. Petrucci, F. Rosetto and G. Spera: arxiv.org/abs/0710.5115 (Annales Fond. de Broglie, in press); F. Cardone, G. Cherubini and A. Petrucci: *Phys. Lett. A* **373**, 862 (2009); F. Cardone, A. Carpinteri and G. Lacidogna: *Phys. Lett. A* **373**, 4158 (2009); F. Cardone, R. Mignani and W. Perconti: *New Advances Phys.* **3**, 7 (2009).